

# The Local Regularity Theory for the Navier–Stokes Equations Near the Boundary

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## Abstract

This is an expository paper on the theory of local regularity for weak solutions to the non-stationary 3D Navier-Stokes equations near the boundary of a domain.

## 1 Introduction

The main problem of the modern mathematical hydrodynamics is the global well-posedness of the 3D Navier-Stokes equations, i.e. the global existence of a unique solution, corresponding to a given smooth divergent-free initial data. There are two main directions in the study of this problem. One can try to improve local well-posedness results, see classical papers [17] and [15] and many others on local well-posedness, or to show that a global weak solution, introduced essentially in [17] and [11] and called the weak Leray-Hopf solution, to the corresponding initial boundary value problem is in fact unique. On the other hand, the second aim can be achieved by proving regularity of weak solutions. Indeed, it is well-known, since the celebrated paper [17] has been published, that smoothness of weak solutions implies their uniqueness in the class of weak Leray-Hopf solutions. Here, we are going to discuss regularity of weak solutions keeping in mind that it is one of possible ways to attack the main problem on the global well-posedness. Our approach is quite typical for PDE's theory and in a sense local. The latter means that we have a solution to the Navier-Stokes system with a “finite energy” in a canonical parabolic domain and try to show that it is smoother in subdomains. The result depends on assumptions imposed on the pressure. Our choice of the class for the pressure is motivated by the linear theory and reflects the fact the whole Navier-Stokes (or Stokes) problem is not quite local because of the incompressibility condition.

Given a space-time point  $z_0 = (x_0, t_0)$ , the canonical domain is going to be a parabolic cylinder  $Q(z_0, R) := B(x_0, R) \times ]t_0 - R^2, t_0[$  if the local interior regularity is studied, or parabolic half-cylinder  $Q^+(z_0, R) := B^+(x_0, R) \times ]t_0 - R^2, t_0[$  if the local

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boundary regularity is under consideration. Here  $B(x_0, R)$  denotes a ball in  $\mathbb{R}^3$  of radius  $R$  centered at a point  $x_0$ , and  $B^+(x_0, R)$  is a half-ball  $B(x_0, R) \cap (x_0 + \mathbb{R}_+^3)$ , and  $\mathbb{R}_+^3 := \{x = (x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_3 > 0\}$ .

Since the Navier-Stokes system is invariant under the scaling transformation

$$u^R(y, s) = Ru(x_0 + Ry, t_0 + R^2s), \quad p^R(y, s) = R^2p(x_0 + Ry, t_0 + R^2s), \quad (1.1)$$

the problem of local regularity of weak solutions in a neighborhood of a point  $z_0$  can be reduced to a model problem in some fixed domain, say,  $Q = B \times ]-1, 0[$  in the internal case or  $Q^+ = B^+ \times ]-1, 0[$  in the boundary case. Here,  $B$  is the unite ball of  $\mathbb{R}^3$  centered at the origin and  $B^+ := B \cap \mathbb{R}_+^3$ .

In contrast to equations of parabolic type, the local smoothing for the Navier-Stokes system has some special features. In particular, in the local setting, weak solutions may be not infinitely smooth in subdomains despite the right-hand side is infinitely smooth there. So, there might be a limiting smoothness which can be achieved locally. The roots of this phenomena lay in the linear theory which we discuss detailly in Section 2.

Typical results in the known local regularity theory have the form of the so-called  $\varepsilon$ -regularity conditions. For the Navier-Stokes system in the canonical domains,  $\varepsilon$ -regularity conditions ensure Hölder continuity of the velocity field around the origin provided a certain integral quantity of a solution over the above domain is sufficiently small.

The first results on the local regularity for the 3D Navier-Stokes equations belong to Scheffer [22]. Scheffer considered a special class of weak solutions to the Cauchy problem that satisfy a *local energy inequality*. Motivated by this observation, later on, Caffarelli-Kohn-Nirenberg introduced the so-called *suitable weak solutions* that are just solutions to the Navier-Stokes system with certain reasonable properties. That was a great step towards a complete local setting. By a definition, a suitable weak solution  $u$  and  $p$  is such that the velocity  $u$  belongs to the energy class, pressure is an integrable function (with a certain exponent of integrability determined by the linear theory and by the integrability of the convective term),  $u$  and  $p$  are assumed to satisfy the Navier-Stokes system in the sense of distributions and the local energy inequality. We will explore the definition of suitable weak solutions introduced in [18], see also [16]:

**Definition 1.1** *We say that a pair of functions  $u$  and  $p$  is a suitable weak solution to the Navier-Stokes system in  $Q$  if*

- $u \in L_{2,\infty}(Q) \cap W_2^{1,0}(Q)$ ,  $p \in L_{\frac{3}{2}}(Q)$
- $u$  and  $p$  satisfy the Navier-Stokes system in  $Q$  in the sense of distributions
- for a.a.  $t \in ]-1, 0[$ , the pair  $u$  and  $p$  satisfies the local energy inequality in  $Q$

$$\begin{aligned} & \int_B \zeta(x, t) |u(x, t)|^2 dx + 2 \int_{-1}^t \int_B \zeta |\nabla u|^2 dx dt \leq \\ & \leq \int_{-1}^t \int_B |u|^2 (\partial_t \zeta + \Delta \zeta) dx dt + \int_{-1}^t \int_B u \cdot \nabla \zeta (|u|^2 + 2p) dx dt \end{aligned}$$

for any non-negative test function  $\zeta \in C^\infty(\mathbb{R}^3 \times \mathbb{R})$  vanishing near the parabolic boundary  $\partial'Q := (\partial B \times ]-1, 0[) \cup (B \times \{t = -1\})$ .

Here we denote by  $L_s(Q)$  the Lebesgue space of functions integrable over  $Q$  with the exponent  $s \in [1, +\infty]$ ;  $W_s^{1,0}(Q) := \{u \in L_s(Q) \mid \nabla u \in L_s(Q)\}$ , where  $\nabla u$  denotes the gradient of  $u$  with respect to spatial variables;  $L_{2,\infty}(Q) := L_\infty(-1, 0; L_2(B))$ .

This class of local solutions appears in the global setting. Indeed, it is not so difficult to show that, in a given space-time domain, among all weak Leray-Hopf solutions, corresponding to a given initial data, there exists at least one that is a suitable weak solution in any parabolic cylinder  $Q(z_0, R)$  belonging to the space-time domain, see [4] and [16].

The significant contribution into the local regularity theory for the Navier-Stokes equation has been made by Caffarelli-Kohn-Nirenberg. They showed that the set of all singular points is very small in the following sense: the one dimensional parabolic Hausdorff measure of this set is equal to zero. This is a consequence of the Caffarelli-Kohn-Nirenberg  $\varepsilon$ -regularity condition reading that there exists an absolute constant  $\varepsilon_0$  such that, for any suitable weak solution  $u$  and  $p$  in  $Q$  with

$$\sup_{r < 1} \frac{1}{r} \int_{Q(r)} |\nabla u|^2 \, dxdt < \varepsilon_0, \quad (1.2)$$

the velocity field  $u$  is essentially bounded near the origin (actually it is Hölder continuous, as it was shown later in [16]). Here and in what follows we denote  $Q(r) := Q(0, r)$ ,  $Q^+(r) := Q^+(0, r)$  etc.

Among various  $\varepsilon$ -regularity conditions, see, for example, papers [4], [18], [16], [30], and many others, we would like to point out the following one: there exists an absolute constant  $\varepsilon_1 > 0$  such that, for any suitable weak solution  $u$  and  $p$  in  $Q$ , satisfying in addition the inequality

$$\int_Q \left( |u|^3 + |p|^{\frac{3}{2}} \right) \, dxdt < \varepsilon_1, \quad (1.3)$$

the velocity field  $u$  is Hölder continuous in the completion of the set  $Q(\frac{1}{2})$ . In the present paper, condition (1.3) and its boundary analogue are called *the basic  $\varepsilon$ -regularity conditions*. The basic  $\varepsilon$ -regularity condition is remarkable as many other  $\varepsilon$ -regularity conditions, including CKN-condition (1.2), can be derived from this one, see, for example, [4], [18], and [16]. In what follows, we are going to deal with the boundary analogue of condition (1.3).

Now, let us review known results on local regularity up to the spatial boundary for weak solutions to the Navier-Stokes system. In this case, we complement the Navier-Stokes equation with the non-slip boundary condition for the velocity field. We focus ourselves mostly on the explanation of what happens with weak solutions to the Navier-Stokes system in the canonical domain  $Q^+$  with the boundary condition on the flat part of semi-ball  $B^+$ , i.e.,

$$u|_{x_3=0} = 0.$$

An analogue of  $\varepsilon$ -regularity condition (1.2) for boundary points has been proven in [24]. The proof is typical for PDE's system and based contradiction arguments. Later on, in [25], the same sufficient regularity condition was proved directly, which made it possible in principle to estimate the size of all constants in the corresponding assumptions.

The latest version of the definition of suitable weak solutions is as follows.

**Definition 1.2** We say that a pair of functions  $u$  and  $p$  is a boundary suitable weak solution to the Navier-Stokes system in  $Q^+$  if

- $u \in L_{2,\infty}(Q^+) \cap W_2^{1,0}(Q^+)$ ,  $p \in L_{\frac{3}{2}}(Q^+)$
- $u|_{x_3=0} = 0$  in the sense of traces
- $u$  and  $p$  satisfy the Navier-Stokes system in  $Q^+$  in the sense of distributions
- for a.a.  $t \in ]-1, 0[$ , the pair  $u$  and  $p$  satisfies the local energy inequality in  $Q^+$

$$\begin{aligned} & \int_{B^+} \zeta(x, t) |u(x, t)|^2 dx + 2 \int_{-1}^t \int_{B^+} \zeta |\nabla u|^2 dx dt \leq \\ & \leq \int_{-1}^t \int_{B^+} |u|^2 (\partial_t \zeta + \Delta \zeta) dx dt + \int_{-1}^t \int_{B^+} u \cdot \nabla \zeta (|u|^2 + 2p) dx dt \end{aligned}$$

for any non-negative test function  $\zeta \in C^\infty(\mathbb{R}^3 \times \mathbb{R})$  vanishing near the parabolic boundary  $\partial' Q$ .

However, in [24], [25], [32], the definition of suitable weak solutions is different. It is supposed there in addition that the second spatial derivatives and the first derivative in time of the velocity field and the gradient of the pressure must exist as integrable functions in  $Q^+$ :

$$u \in W_{s,l}^{2,1}(Q^+), \quad p \in W_{s,l}^{1,0}(Q^+) \quad \text{for some } s, l \in ]1, +\infty[ \quad \text{such that} \quad \frac{3}{s} + \frac{2}{l} \geq 4. \quad (1.4)$$

Here  $L_{s,l}(Q^+)$  is the anisotropic Lebesgue space equipped with the norm

$$\|f\|_{L_{s,l}(Q^+)} := \left( \int_{-1}^0 \left( \int_{B^+} |f(x, t)|^s dx \right)^{l/s} dt \right)^{1/l},$$

and we use the following notation for the functional spaces:

$$\begin{aligned} W_{s,l}^{1,0}(Q^+) &\equiv L_t(-1, 0; W_s^1(B^+)) = \{ u \in L_{s,l}(Q^+) : \nabla u \in L_{s,l}(Q^+) \}, \\ W_{s,l}^{2,1}(Q^+) &= \{ u \in W_{s,l}^{1,0}(Q^+) : \nabla^2 u, \partial_t u \in L_{s,l}(Q^+) \}, \end{aligned}$$

and the following notation for the norms:

$$\begin{aligned} \|u\|_{W_{s,l}^{1,0}(Q^+)} &= \|u\|_{L_{s,l}(Q^+)} + \|\nabla u\|_{L_{s,l}(Q^+)}, \\ \|u\|_{W_{s,l}^{2,1}(Q^+)} &= \|u\|_{W_{s,l}^{1,0}(Q^+)} + \|\nabla^2 u\|_{L_{s,l}(Q^+)} + \|\partial_t u\|_{L_{s,l}(Q^+)}. \end{aligned}$$

A particular choice of exponents  $s, l$  is a matter of convenience and it can be made in various ways. For example, in [24] and in [32], it is assumed that  $s = 9/8$ ,  $l = 3/2$ , in [25]  $s = l = 5/4$  and  $s = 15/14$ ,  $l = 5/3$ . It should be mentioned that the choice of  $s, l$  is always motivated by the following general idea: if we take the Leray-Hopf solution  $u$  of the initial-boundary value problem for the Navier-Stokes equation in some domain and interpret the convective term  $(u \cdot \nabla)u$  as a right-hand side of the initial-boundary value problem for the Stokes system then extra conditions (1.4) follows from the coercive

estimates of the linear theory and from the uniqueness theorem for the Stokes problem in the class of weak solutions. This scheme works well in the case of interior regularity, while, for the boundary regularity case, its realization encounters some difficulties. Nevertheless, as it is shown in [26], the scheme works up to the boundary as well and extra assumptions (1.4) are simply superfluous.

As it has been already mentioned, there is an essential difference between local interior regularity and local boundary regularity even for the Stokes system. One of the consequences of such an observation is that the further smoothing in a neighborhood of regular points might happen differently. In a neighborhood of an interior regular point, a suitable weak solution has all the spatial derivatives that are Hölder continuous, while, in a neighborhood of boundary regular point, the spatial gradient is not necessary to be even bounded, see [13] and [33].

Besides conditions (1.2), (1.3),  $\varepsilon$ -regularity theory provides many other sufficient conditions of local regularity of suitable weak solutions to the Navier-Stokes equations, see, for example, papers [12], [28], [29], [30], and references in them. Most of these conditions are stated in terms of so-called *scale invariant functionals*. Some examples of such functionals (in the case of the flat part of the boundary) are as follows:

$$\begin{aligned} A(u, r) &= \sup_{t \in (-r^2, 0)} \left( \frac{1}{r} \int_{B^+(r)} |u(x, t)|^2 dx \right)^{1/2}, \\ C(u, r) &= \left( \frac{1}{r^2} \int_{Q^+(r)} |u(x, t)|^3 dx dt \right)^{1/3}, \quad E(u, r) = \left( \frac{1}{r} \int_{Q^+(r)} |\nabla u(x, t)|^2 dx dt \right)^{1/2} \end{aligned} \quad (1.5)$$

“Scale invariance” means that if  $F(u, r)$  is one of these functionals and  $u^R$  and  $p^R$  are functions obtained from  $u$  and  $p$  by formulas (1.1) with  $x_0 = 0$ ,  $t_0 = 0$ , then

$$F(u^R, 1) = F(u, R), \quad \forall R > 0.$$

The functionals  $A(u, r)$ ,  $C(u, r)$ ,  $E(u, r)$  possess the following property: boundedness of one of them, i.e.

$$\min \left\{ \sup_{r < 1} A(u, r), \sup_{r < 1} C(u, r), \sup_{r < 1} E(u, r) \right\} < +\infty,$$

implies boundedness of all others:

$$\max \left\{ \sup_{r < 1} A(u, r), \sup_{r < 1} C(u, r), \sup_{r < 1} E(u, r), \sup_{r < 1} D(p, r) \right\} < +\infty. \quad (1.6)$$

Here,  $D(p, r)$  is a functional

$$D(p, r) = \left( \frac{1}{r^2} \int_{Q^+(r)} |p(x, t)|^{3/2} dx dt \right)^{2/3},$$

which is also invariant under the scaling transformation of  $p$  according to (1.1). Statement (1.6) has been proven in [28] in the internal case. Later on, in [19], it was generalized to the boundary case.

One of the basic principles in the  $\varepsilon$ -regularity theory for the Navier-Stokes equations reads: if at least one of the scale invariant functionals is small uniformly with respect to all  $r \in ]0, 1[$ , i.e.

$$\sup_{r < 1} F(u, r) < \varepsilon_0, \quad (1.7)$$

then the origin is a regular point of the velocity field  $u$  (i.e.  $u$  is Hölder continuous near the origin). This statement has been rigorously proven by many authors for various types of scale invariant functionals, see, for example, references in [12], [10], [29], [30]. In our paper, we shall mention only those contributions that are concerned with the boundary case. In [24], [25], the boundary regularity up to the flat part of the boundary has been proven if  $F(u, r)$  is one of functionals in (1.5).

There is one more example of scale invariant functionals, which we call the Ladyzhenskaya-Prodi-Serrin-type functional (LPS-functional):

$$M_{s,l}(u, r) := \|u\|_{L_{s,l}(Q^+(r))} = \left( \int_{-r^2}^0 \left( \int_{B^+(r)} |u(x, t)|^s dx \right)^{l/s} dt \right)^{1/l}, \quad s, l \in ]1, +\infty[$$

Comparably with functionals defined in (1.5) LPS functionals possess two additional properties. First, it is monotone with respect to  $r$  and hence if it is finite for some particular  $r_0 > 0$  then it is uniformly bounded for all  $r \in ]0, r_0[$ . And second, if  $s, l > 1$ , then LPS functionals are absolutely continuous functions of a domain, i.e.

$$M_{s,l}(u, r) \rightarrow 0 \quad \text{as} \quad r \rightarrow 0. \quad (1.8)$$

So,  $\varepsilon$ -regularity theory developed above provides a simple proof of the following conditional result for a boundary suitable weak solution  $u$  to the Navier-Stokes equations in  $Q^+$ : if  $s > 3$  and

$$\frac{3}{s} + \frac{2}{l} = 1 \quad \text{and} \quad M_{s,l}(u, 1) < +\infty, \quad (1.9)$$

then  $u$  is regular near the origin. For a different approach, we refer to paper [12].

Note that formally the first inequality in (1.9) allows the following combination of parameters:  $s = 3$ ,  $l = +\infty$ , i.e. one can ask the question about local regularity of solutions to the Navier-Stokes system belonging to the class  $L_{3,\infty}(Q^+) := L_\infty(-1, 0; L_3(B^+))$  which we call  $L_{3,\infty}$ -solutions. It is known (see [14]) that the initial boundary value problem for the Navier-Stokes equation is locally well-posed on  $L_3$ -space. But the method that we used to prove regularity weak solution with finite LPS functionals in the case of  $s > 3$ ,  $l < +\infty$  can not be extended to  $L_{3,\infty}$  case as the functional  $M_{s,l}(u, r)$  with  $s = 3$ ,  $l = +\infty$  in general does not possess the property (1.8).

The proof of regularity of  $L_{3,\infty}$ -solutions to the Navier-Stokes system requires development of the completely new approach which is based on the backward-in-time uniqueness for the heat operator with lower order coefficients in the compliment to a ball or even in a half space. This method has been introduced in [6] and then developed in [5] and in [7] in order to prove the interior regularity of  $L_{3,\infty}$ -solutions. Later on, the same type of results was extended to the boundary case of  $L_{3,\infty}$ -solutions. It has been shown in [31] that:

$$u \in L_{3,\infty}(Q^+) \quad \implies \quad u \in C^{\alpha, \frac{\alpha}{2}}(\bar{Q}^+(\frac{1}{2}))$$

Here  $C^{\alpha, \frac{\alpha}{2}}(\bar{Q}^+(r))$  denotes the space of functions which are Hölder continuous with the exponent  $\alpha > 0$  with respect to the usual parabolic metric.

The paper is organized as follows. In Section 2 we discuss the local smoothness of weak solutions to the linear Stokes problem near the boundary. In Section 3 we present

a proof of the basic  $\varepsilon$ -regularity condition for boundary suitable weak solutions to the Navier-Stokes equations near a flat part of the boundary. Finally, in Section 4 we give a brief overview of known results on the local regularity theory for the Navier-Stokes equations in a domain with a curvilinear boundary.

## 2 Linear Theory

As it is mentioned in the introduction, weak solutions to the non-stationary Stokes system

$$\begin{cases} \partial_t u - \Delta u + \nabla p = 0 \\ \operatorname{div} u = 0 \end{cases} \quad \text{in } Q \quad (2.1)$$

locally are not necessary smooth. A simple example of a non-smooth solution to (2.1) is as follows:

$$u(x, t) = \varphi(t) \nabla h(x), \quad p(x, t) = -\varphi'(t) h(x),$$

where  $h$  is a scalar harmonic function in spatial variables and  $\varphi$  is an arbitrary function of  $t$  having limited smoothness. The same effect takes place in non-linear case and has been pointed out by J. Serrin in [35].

Nevertheless, Stokes system (2.1) has the property of infinite smoothing of weak solutions with respect to spatial variables in internal points of a domain:

**Theorem 2.1** *Assume  $s \in ]1, +\infty[$ ,  $l \in ]1, 2[$ , and  $u \in W_{s,l}^{1,0}(Q)$ ,  $p \in L_{s,l}(Q)$  satisfy (2.1) in  $Q$  in the sense of distributions. Then for any  $k = 0, 1, \dots$ , we have  $\nabla^k u \in C^{\alpha, \frac{\alpha}{2}}(\bar{Q}(\frac{1}{2}))$  with  $\alpha = 2 - \frac{2}{l}$ .*

Surprisingly, the analog of Theorem 2.1 is not valid if we consider weak solutions to the Stokes system near the boundary

$$\begin{cases} \partial_t u - \Delta u + \nabla p = f \\ \operatorname{div} u = 0 \\ u|_{x_3=0} = 0. \end{cases} \quad \text{in } Q^+ \quad (2.2)$$

Actually, in contrast to the internal case, the first spatial gradient of a weak solution to system (2.2) is not necessary bounded up to the boundary, i.e. there exist functions  $u \in W_2^{1,0}(Q^+)$ ,  $p \in L_{\frac{3}{2}}(Q^+)$  which satisfy system (2.2) with  $f \equiv 0$  in  $Q^+$  in the sense of distributions,  $u$  satisfy the boundary condition in the sense of traces but

$$\nabla u \notin L_\infty(Q^+(r)) \quad (2.3)$$

for any  $0 < r \leq 1/2$ . The first counterexample of this kind has been constructed by Kang in [13]. Later Seregin and Sverak in [33] simplified his construction significantly. Here, we explain the counter-example, following to [33]:

**Example 2.1** Assume  $\varphi(t)$  is an arbitrary function of  $t$  variable and let  $h : \mathbb{R}_+ \times ]-2, 0[ \rightarrow \mathbb{R}$ ,  $h = h(x, t)$  be a solution to the following initial boundary value problem for the 1D heat equation in a half-line:

$$\begin{cases} \frac{\partial h}{\partial t} - \frac{\partial^2 h}{\partial x^2} = \varphi(t) & \text{in } \mathbb{R}_+ \times ]-2, 0[ \\ h|_{t=-2} = 0, & h|_{x=0} = 0. \end{cases}$$

Let  $u : \mathbb{R}_+^3 \times ]-2, 0[ \rightarrow \mathbb{R}^3$ ,  $p : \mathbb{R}_+^3 \times ]-2, 0[ \rightarrow \mathbb{R}$  be functions representing the following shear flow along  $x_1$ -axe:

$$u(x, t) := (h(x_3, t), 0, 0), \quad p(x, t) := -\varphi(t)x_1$$

Then functions  $u$  and  $p$  satisfy (formally) both the Stokes and the Navier-Stokes systems in  $Q^+$ . Moreover, if we assume that  $\alpha \in ]\frac{1}{3}, \frac{1}{2}[$  and take

$$\varphi(t) = \frac{1}{|t|^{1-\alpha}}$$

then  $u \in W_2^{1,0}(Q^+)$  and  $p \in L_{\frac{3}{2}}(Q^+)$ . Functions  $u$  and  $p$  satisfy equations (2.2) with  $f = 0$  in the sense of distributions and the boundary condition in the sense of traces as well. However,  $\nabla u$  is unbounded in any neighborhood of the origin. In particular, (2.3) holds.

Example 2.1 shows that, in contrast to the internal case, the Stokes system does not possess the property of significant improvement of the regularity of weak solutions up to the boundary. This is a serious obstacle which makes the theory of boundary regularity for the Navier-Stokes equations different from the analogues theory in the internal case. The natural question that arises is what is the optimal regularity of weak solutions to the Stokes system up to the boundary in the local set-up. A certain answer to this question has been given in [23], where Hölder continuity of the velocity field  $u$  up to the flat part of the boundary has been established for *strong solutions* to (2.2). But before we switch to the discussion of this result let us introduce some terminology to explain the difference between weak and strong solutions to (2.2).

**Definition 2.1** Assume  $1 < s, l < +\infty$  and  $f \in L_l(-1, 0; W_s^{-1}(B^+))$ . We say that functions  $u$  and  $p$  are a weak solution of system (2.2), if they belong to the spaces

$$u \in W_{s,l}^{1,0}(Q^+), \quad p \in L_{s,l}(Q^+),$$

$u$  and  $p$  satisfy (2.2) in the sense of distributions and  $u$  satisfies the boundary condition in the sense of traces.

Note that, for any weak solution  $u$  and  $p$  to system (2.2),  $\partial_t u \in L_l(-1, 0; W_s^{-1}(B^+))$  and the following estimate holds:

$$\begin{aligned} \|\partial_t u\|_{L_l(-1,0;W_s^{-1}(B^+))} &\leq \\ &\leq C \left( \|f\|_{L_l(-1,0;W_s^{-1}(B^+))} + \|u\|_{W_{s,l}^{1,0}(Q^+)} + \|p\|_{L_{s,l}(Q^+)} \right) \end{aligned} \quad (2.4)$$

Here  $W_s^{-1}(B^+)$  is the space dual to  $\mathring{W}_s^1(B^+)$  and

$$\|u\|_{L_l(-1,0;W_s^{-1}(B^+))} = \left( \int_{-1}^0 \|u(\cdot, t)\|_{W_s^{-1}(B^+)}^l dt \right)^{1/l}.$$



**Definition 2.2** Assume  $1 < s, l < +\infty$  and  $f \in L_{s,l}(Q^+)$ . We say that functions  $u$  and  $p$  are a strong solution to (2.2) if they are a weak solution to (2.2) and

$$u \in W_{s,l}^{2,1}(Q^+), \quad p \in W_{s,l}^{1,0}(Q^+).$$

The idea of showing Hölder continuity of strong solutions proposed in [23] is as follows. We first show that strong solutions satisfy the usual local version of coercive estimate given by the following theorem:

**Theorem 2.2** Suppose  $s, l \in ]1, \infty[$ . For any  $f \in L_{s,l}(Q^+)$  and for any strong solution  $u \in W_{s,l}^{2,1}(Q^+)$ ,  $p \in W_{s,l}^{1,0}(Q^+)$  to system (2.2) in  $Q^+$ , the following local estimate holds:

$$\begin{aligned} & \|u\|_{W_{s,l}^{2,1}(Q^+(\frac{1}{2}))} + \|\nabla p\|_{L_{s,l}(Q^+(\frac{1}{2}))} \leq \\ & \leq C \left( \|f\|_{L_{s,l}(Q^+)} + \|\nabla u\|_{L_{s,l}(Q^+)} + \|p\|_{L_{s,l}(Q^+)} \right) \end{aligned} \quad (2.5)$$

with a positive constant  $C$ , depending only on  $s, l$ .

In contrast to the interior case, this is not a trivial statement. The first proof given in [23] has been based on duality arguments and inspired by paper [37]. In the present paper, we show that it can be deduced from more general statement proved in [8]. But before explaining our approach let us briefly described the main idea of getting Hölder continuity. Unlike it has been done in the internal case, in the boundary case we can not get the result by gaining more derivatives in space variables. In fact, this is even impossible because of the counterexample in Example 2.1. But what we can do is to gain more integrability in space and to apply certain bootstrap arguments. So, we show the following:

**Theorem 2.3** Suppose  $s, l, m \in ]1, \infty[$ ,  $m \geq s$ . For any  $f \in L_{m,l}(Q^+)$  and for any strong solution  $u \in W_{s,l}^{2,1}(Q^+)$ ,  $p \in W_{s,l}^{1,0}(Q^+)$  to (2.2) in  $Q^+$ , we have

$$u \in W_{m,l}^{2,1}(Q^+(\frac{1}{2})), \quad \nabla p \in L_{m,l}(Q^+(\frac{1}{2}))$$

and the following local estimate holds:

$$\begin{aligned} & \|u\|_{W_{m,l}^{2,1}(Q^+(\frac{1}{2}))} + \|\nabla p\|_{L_{m,l}(Q^+(\frac{1}{2}))} \leq \\ & \leq C \left( \|f\|_{L_{m,l}(Q^+)} + \|\nabla u\|_{L_{s,l}(Q^+)} + \|p\|_{L_{s,l}(Q^+)} \right) \end{aligned} \quad (2.6)$$

with a positive constant  $C$ , depending only on  $s, l, m$ .

If the exponent  $m$  in Theorem 2.3 is sufficiently large then the Hölder continuity of  $u$  follows from the imbedding theorems for the anisotropic Sobolev spaces. Namely, the following result is true:

**Theorem 2.4** Suppose  $s, l, m \in ]1, \infty[$ ,  $m > \frac{3l}{2(l-1)}$ . For any  $f \in L_{m,l}(Q^+)$  and for any strong solution  $u \in W_{s,l}^{2,1}(Q^+)$ ,  $p \in W_{s,l}^{1,0}(Q^+)$  to system (2.2), we have  $u \in C^{\beta, \frac{\beta}{2}}(\bar{Q}^+(\frac{1}{2}))$  with  $\beta = 2 - \frac{3}{m} - \frac{2}{l}$  and the following local estimate holds:

$$\|u\|_{C^{\beta, \frac{\beta}{2}}(\bar{Q}^+(\frac{1}{2}))} \leq C \left( \|f\|_{L_{m,l}(Q^+)} + \|\nabla u\|_{L_{s,l}(Q^+)} + \|p\|_{L_{s,l}(Q^+)} \right), \quad (2.7)$$

where a positive constant  $C$  depends only on  $s, l, m$ .

The important part is to show that any weak solution to (2.2) is actually strong one at least locally (here we use the terminology introduced in our Definitions 2.1 and 2.2). For the interior case, the corresponding claim is known (see details, for example, in [16]). For the boundary case, the analogous statement has been proven in [26] relatively recently. Here, our proof follows to [36]. It is some modification of the approach in [26], which can be used in more general situations. So, the result is as follows:

**Theorem 2.5** *Assume  $s, l \in ]1, \infty[$ . Then, for any  $f \in L_{s,l}(Q^+)$  and for any weak solution  $u \in W_{s,l}^{1,0}(Q^+)$ ,  $p \in L_{s,l}(Q^+)$  to (2.2) in  $Q^+$ , the following statements*

$$u \in W_{s,l}^{2,1}(Q^+(\tfrac{1}{2})), \quad p \in W_{s,l}^{1,0}(Q^+(\tfrac{1}{2}))$$

*hold and functions  $u$  and  $p$  are a strong solution to system (2.2) in the half-cylinder  $Q^+(\tfrac{1}{2})$ .*

As our example shows, regularity results up to the boundary described by the above statements are in a sense optimal.

Now we come to the detailed proofs of the results above.

**Proof of Theorem 2.2:** The proof presented bellow is borrowed from [8]. We reproduce it here for the sake of completeness. Take arbitrary  $\rho, r$  such that  $\frac{1}{2} \leq \rho < r \leq \frac{9}{10}$ . Consider a cut-off function  $\zeta \in C^\infty(\bar{Q}^+)$  such that

$$\begin{aligned} 0 \leq \zeta \leq 1 \quad \text{in } Q^+, \quad \zeta \equiv 1 \quad \text{in } Q^+(\rho), \quad \zeta \equiv 0 \quad \text{in } Q^+ \setminus Q^+(r), \\ \|\nabla^k \zeta\|_{L_\infty(Q^+)} \leq \frac{C}{(r-\rho)^k}, \quad k = 1, 2, \quad \|\partial_t \nabla^k \zeta\|_{L_\infty(Q^+)} \leq \frac{C}{(r-\rho)^k}, \quad k = 0, 1. \end{aligned}$$

Let  $u$  and  $p$  be a strong solution to system (2.2). Then functions  $v := \zeta u$ ,  $q := \zeta p$  satisfy the following initial-boundary value problem

$$\begin{cases} \partial_t v - \Delta v + \nabla q = \tilde{f} & \text{in } \Omega \times ]-1, 0[, \\ \operatorname{div} v = g & \\ v|_{\partial\Omega \times ]-1, 0[} = 0, \quad v|_{t=-1} = 0. \end{cases} \quad (2.8)$$

where

$$\tilde{f} = \zeta f + u(\partial_t \zeta - \Delta \zeta) - 2(\nabla u) \nabla \zeta + p \nabla \zeta, \quad g = u \cdot \nabla \zeta \quad (2.9)$$

and  $\Omega$  is some smooth canonical domain which is diffeomorphic to a ball and satisfies the inclusions  $B_{9/10}^+ \subset \Omega \subset B^+$ . Applying Theorem 1.1 of [8], we obtain the estimate

$$\begin{aligned} & \|v\|_{W_{s,l}^{2,1}(\tilde{Q})} + \|\nabla q\|_{L_{s,l}(\tilde{Q})} \leq \\ & \leq C \left( \|\tilde{f}\|_{L_{s,l}(\tilde{Q})} + \|g\|_{W_{s,l}^{1,0}(\tilde{Q})} + \|\partial_t g\|_{L_{s,l}(\tilde{Q})}^{1/s} \|\partial_t g\|_{L_l(-1,0;W_s^{-1}(\Omega))}^{1/s'} \right) \end{aligned} \quad (2.10)$$

where we denote  $\tilde{Q} := \Omega \times ]-1, 0[$ . Taking into account that  $\frac{1}{r-\rho} \geq 1$  after routine computations we obtain the estimate

$$\begin{aligned} & \|u\|_{W_{s,l}^{2,1}(Q^+(\rho))}^s \leq C \|f\|_{L_{s,l}(Q^+)}^s \\ & + \frac{C}{(r-\rho)^{2s}} \left( \|u\|_{W_{s,l}^{1,0}(Q^+)}^s + \|p\|_{L_{s,l}(Q^+)}^s + \|\partial_t u\|_{L_l(-1,0;W_s^{-1}(B^+))}^s \right) \\ & + \frac{C}{(r-\rho)^{2s}} \|\partial_t u\|_{L_{s,l}(Q^+(r))}^s \left( \|\partial_t u\|_{L_l(-1,0;W_s^{-1}(B^+))}^{s-1} + \|u\|_{L_{s,l}(Q^+)}^{s-1} \right). \end{aligned} \quad (2.11)$$

Estimating the last term in the right-hand side of (2.11) via the Young inequality  $ab \leq \varepsilon a^s + C_\varepsilon b^{s'}$  we obtain

$$\begin{aligned} \|u\|_{W_{s,l}^{2,1}(Q^+(\rho))}^s &\leq C \|f\|_{L_{s,l}(Q^+)}^s + \varepsilon \|\partial_t u\|_{L_{s,l}(Q^+(r))}^s + \\ &+ \frac{C_\varepsilon}{(r-\rho)^{2ss'}} \left( \|u\|_{W_{s,l}^{1,0}(Q^+)}^s + \|p\|_{L_{s,l}(Q^+)}^s + \|\partial_t u\|_{L_l(-1,0;W_s^{-1}(B^+))}^s \right), \end{aligned}$$

where a constant  $\varepsilon > 0$  can be chosen arbitrary small. By virtue of (2.4), we obtain

$$\|u\|_{W_{s,l}^{2,1}(Q^+(\rho))}^s \leq \varepsilon \|\partial_t u\|_{L_{s,l}(Q^+(r))}^s + \frac{C_\varepsilon}{(r-\rho)^{2ss'}} \left( \|f\|_{L_{s,l}(Q^+)}^s + \|u\|_{W_{s,l}^{1,0}(Q^+)}^s + \|p\|_{L_{s,l}(Q^+)}^s \right). \quad (2.12)$$

Now, let us introduce the monotone function  $\Psi(\rho) := \|u\|_{W_{s,l}^{2,1}(Q^+(\rho))}^s$  and the constant

$$A := C_\varepsilon \left( \|f\|_{L_{s,l}(Q^+)}^s + \|u\|_{W_{s,l}^{1,0}(Q^+)}^s + \|p\|_{L_{s,l}(Q^+)}^s \right).$$

The inequality (2.12) implies that

$$\Psi(\rho) \leq \varepsilon \Psi(r) + \frac{A}{(r-\rho)^\alpha}, \quad \forall \rho, r : R_1 \leq \rho < r \leq R_0, \quad (2.13)$$

for some  $\alpha > 0$  depending only on  $s$ , and for  $R_1 = \frac{1}{2}$ ,  $R_0 = \frac{9}{10}$ . Now we shall take an advantage of the following lemma (which can be easily proved by iterations if one take  $r_k := R_0 - 2^{-k}(R_0 - R_1)$ ):

**Lemma 2.1** *Assume  $\Psi$  is a nondecreasing bounded function which satisfies inequality (2.13) for some  $\alpha > 0$ ,  $A > 0$ , and  $\varepsilon \in ]0, 2^{-\alpha}[$ . Then there exists a constant  $B$  depending only on  $\varepsilon$  and  $\alpha$  such that*

$$\Psi(R_1) \leq \frac{B A}{(R_0 - R_1)^\alpha}.$$

Fixing  $\varepsilon = 2^{-3ss'}$  in (2.12), applying Lemma 2.1 to our function  $\Psi$ , and evaluating  $\nabla p$  from equations (2.2) held a.e. in  $Q^+$ , we derive the estimate (2.5). Theorem 2.2 is proved.  $\square$

Theorems 2.2 together with results of [8] provides us the following proof of Theorem 2.5.

**Proof of Theorem 2.5.** Let  $\rho_m \rightarrow +0$  be an arbitrary sequence. Extend all functions  $u, p, f$  from  $Q^+$  to the set  $B^+ \times \mathbb{R}$  by zero. For any extended function  $u$  denote by  $u^m$  the mollification of the function  $u$  with respect to  $t$  variable:

$$u^m(x, t) := (\omega_{\rho_m} * u)(x, t) \equiv \int_{\mathbb{R}} \omega_{\rho_m}(t - \tau) u(x, \tau) d\tau,$$

where  $\omega_\rho(t) = \frac{1}{\rho} \omega(t/\rho)$ , and  $\omega \in C_0^\infty(-1, 1)$  is a smooth kernel normalized by the identity  $\int_0^1 \omega(t) dt = 1$ .

As  $u \in W_{s,l}^{1,0}(Q^+)$ ,  $p \in L_{s,l}(Q^+)$ ,  $f \in L_{s,l}(Q^+)$  we have

$$\begin{aligned} u^m &\rightarrow u \quad \text{in } W_{s,l}^{1,0}(Q^+), \quad p^m \rightarrow p \quad \text{in } L_{s,l}(Q^+), \\ f^m &\rightarrow f \quad \text{in } L_{s,l}(Q^+). \end{aligned} \quad (2.14)$$

Let us fix arbitrary  $\delta \in ]0, \frac{1}{12}[$ . Then for any  $\rho_m < \delta$  and for any  $\eta \in C^\infty(\bar{Q}^+)$

$$\partial_t(\omega_{\rho_m} * \eta)(x, t) = (\omega_{\rho_m} * \partial_t \eta)(x, t), \quad \forall x \in B^+, t \in ]-1 + \delta, -\delta[.$$

A weak solution  $u$  and  $p$  to system (2.2) obeys the integral identity

$$-\int_{Q^+} u \cdot (\partial_t \eta + \Delta \eta) \, dx dt = \int_{Q^+} (f \cdot \eta + p \operatorname{div} \eta) \, dx dt$$

which holds for all  $\eta \in C^\infty(\bar{Q}^+)$  satisfying conditions

$$\eta|_{\partial B^+ \times ]-1, 0[} = 0, \quad \nabla \eta|_{\partial' B^+ \times ]-1, 0[} = 0, \quad \eta|_{B^+ \times (]-1, -1+\delta[ \cup ]-\delta, 0[)} = 0, \quad (2.15)$$

where  $\partial' B^+ := \{x \in \mathbb{R}^n : |x| = 1, x_n > 0\}$ . Take the test function  $\eta = \omega_{\rho_m} * \tilde{\eta}$ , where  $\tilde{\eta} \in C^\infty(\bar{Q}^+)$  is an arbitrary function satisfying (2.15). Using properties of convolution, we find the identity

$$-\int_{Q^+} u^m \cdot (\partial_t \tilde{\eta} + \Delta \tilde{\eta}) \, dx dt = \int_{Q^+} (f^m \cdot \tilde{\eta} + p^m \operatorname{div} \tilde{\eta}) \, dx dt \quad (2.16)$$

which holds for all  $\tilde{\eta} \in C^\infty(\bar{Q}^+)$  satisfying (2.15).

Let  $\zeta \in C^\infty(\bar{Q}^+)$  be a cut-of function vanishing in  $Q^+ \setminus Q^+(\frac{5}{6})$  and such that  $\zeta \equiv 1$  in  $Q^+(\frac{2}{3})$ . Denote  $v^m := \zeta u^m$ ,  $q^m := \zeta p^m$ . Then from (2.16) we deduce that  $v^m$  and  $q^m$  obey the integral identity

$$-\int_{B^+ \times ]-1, -\delta[} v^m \cdot (\partial_t \eta + \Delta \eta) \, dx dt = \int_{B^+ \times ]-1, -\delta[} (\tilde{f}^m \cdot \eta + q^m \operatorname{div} \eta) \, dx dt$$

for any  $\eta \in C^\infty(\bar{B}^+ \times [-1, -\delta])$  such that  $\eta|_{\partial B^+ \times ]-1, -\delta[} = 0$  and  $\eta|_{B^+ \times \{t=-\delta\}} = 0$ . Here  $\tilde{f}^m$  and  $g^m$  are determined by formulas (2.9) with  $u$ ,  $p$  and  $f$  replaced by  $u^m$ ,  $p^m$  and  $f^m$  respectively.

Assume  $\Omega \subset \mathbb{R}^3$  is a smooth domain such that  $B^+(\frac{5}{6}) \subset \Omega \subset B^+$  and denote  $\tilde{Q} := \Omega \times ]-1, 0[$ . As functions  $g^m$  are smooth with respect to  $t$ , we obtain from Theorem 1.1 of [8] that, for any  $m \in \mathbb{N}$ , there exists a strong solution  $\tilde{v}^m \in W_{s,l}^{2,1}(\tilde{Q})$ ,  $\tilde{q}^m \in W_{s,l}^{1,0}(\tilde{Q})$  to the problem

$$\begin{cases} \partial_t \tilde{v}^m - \Delta \tilde{v}^m + \nabla \tilde{q}^m = \tilde{f}^m \\ \operatorname{div} \tilde{v}^m = g^m \end{cases} \quad \text{in } \tilde{Q}, \quad (2.17)$$

$$\tilde{v}^m|_{\partial \Omega \times ]-1, 0[} = 0, \quad \tilde{v}^m|_{t=-1} = 0.$$

Note that  $\zeta \equiv 1$  in  $Q^+(\frac{2}{3})$  and hence  $g^m \equiv 0$  in  $Q^+(\frac{2}{3})$ . So, functions  $\tilde{v}^m$  and  $\tilde{q}^m$  satisfy all assumptions of Theorem 2.2 in  $Q^+(\frac{2}{3})$  and, hence, by its obvious modification, we have the estimate

$$\begin{aligned} & \|\tilde{v}^m\|_{W_{s,l}^{2,1}(Q^+(\frac{1}{2}))} + \|\nabla \tilde{q}^m\|_{L_{s,l}(Q^+(\frac{1}{2}))} \leq \\ & \leq C \left( \|\tilde{f}^m\|_{L_{s,l}(Q^+(\frac{2}{3}))} + \|\tilde{v}^m\|_{W_{s,l}^{1,0}(Q^+(\frac{2}{3}))} + \|\tilde{q}^m\|_{L_{s,l}(Q^+(\frac{2}{3}))} \right) \end{aligned} \quad (2.18)$$

where a constant  $C$  depends neither on  $m$  nor on  $\delta$ .

Since every strong solution of the Stokes system is a weak one,  $\tilde{v}^m$  and  $\tilde{q}^m$  satisfy the integral identity

$$- \int_{\tilde{Q}} \tilde{v}^m \cdot (\partial_t \eta + \Delta \eta) \, dx dt = \int_{\tilde{Q}} (\tilde{f}^m \cdot \eta + \tilde{q}^m \operatorname{div} \eta) \, dx dt$$

for all  $\eta \in C^\infty(\overline{\tilde{Q}})$  such that  $\eta|_{\partial\Omega \times ]-1, 0[} = 0$  and  $\eta|_{\Omega \times \{t=0\}} = 0$ . Hence the differences  $w^m := v^m - \tilde{v}^m$ ,  $\pi^m := q^m - \tilde{q}^m$  are a weak solution to the Stokes system in  $\Omega \times ]-1, -\delta[$ , satisfying the identities

$$\begin{aligned} \operatorname{div} w^m &= 0 \quad \text{a.e. in } \Omega \times ]-1, -\delta[, \\ - \int_{\Omega \times ]-1, -\delta[} w^m \cdot (\partial_t \eta + \Delta \eta) \, dx dt &= \int_{\Omega \times ]-1, -\delta[} \pi^m \operatorname{div} \eta \, dx dt, \end{aligned} \quad (2.19)$$

for any  $\eta \in W_{s', l'}^{2,1}(\Omega \times ]-1, -\delta[)$  such that  $\eta|_{\partial\Omega \times ]-1, -\delta[} = 0$  and  $\eta|_{\Omega \times \{t=-\delta\}} = 0$ . Denote  $\varkappa = \min\{s, l\} > 1$ . As  $u^m, \tilde{u}^m \in L_{s, l}(\tilde{Q})$  and  $q^m, \tilde{q}^m \in L_{s, l}(\tilde{Q})$  we have  $w^m = v^m - \tilde{v}^m \in L_\varkappa(\tilde{Q})$  and  $\pi^m = q^m - \tilde{q}^m \in L_\varkappa(\tilde{Q}^+)$ . Hence  $|w^m|^{\varkappa-2} w^m \in L_{\varkappa'}(\tilde{Q})$ , and using results of [37] we can find functions  $\eta \in W_{\varkappa'}^{2,1}(\Omega \times ]-1, -\delta[)$  and  $\kappa \in W_{\varkappa'}^{1,0}(\Omega \times ]-1, -\delta[)$  such that

$$\begin{cases} \partial_t \eta + \Delta \eta + \nabla \kappa = |w^m|^{\varkappa-2} w^m, & \text{in } \Omega \times ]-1, -\delta[, \\ \operatorname{div} \eta = 0, \\ \eta|_{\partial\Omega \times ]-1, -\delta[} = 0, & \eta|_{t=-\delta} = 0. \end{cases}$$

Substituting this  $\eta$  as a test function into identity (2.19) we obtain  $w^m = 0$  in  $\Omega \times ]-1, -\delta[$ . Hence  $v^m = \tilde{v}^m \in W_{s, l}^{2,1}(\Omega \times ]-1, -\delta[)$ . From (2.19) we obtain

$$\int_{\Omega \times ]-1, -\delta[} \pi^m \operatorname{div} \eta \, dx dt = 0, \quad \forall \eta \in L_{l'}(-1, -\delta; \mathring{W}_{s'}^1(\Omega)). \quad (2.20)$$

Correcting, if necessary, the function  $\tilde{q}^m$  by a constant, we can assume that  $\int_{\tilde{Q}} \pi^m \, dx = 0$  for a.e.  $t \in ]-1, -\delta[$ . As  $\pi^m \in L_\varkappa(\Omega)$  for a.e.  $t \in ]-1, -\delta[$ , we have  $|\pi^m|^{\varkappa-2} \pi^m \in L_{\varkappa'}(\Omega)$  for a.e.  $t \in ]-1, -\delta[$ . Using results of [2] for a.e.  $t$  we can find  $\eta(\cdot, t) \in \mathring{W}_{\varkappa'}^1(\Omega)$  such that

$$\begin{cases} \operatorname{div} \eta = |\pi^m|^{\varkappa-2} \pi^m - (|\pi^m|^{\varkappa-2} \pi^m)_\Omega, & \text{a.e. } t \in ]-1, -\delta[, \\ \|\eta\|_{W_{\varkappa'}^1(\Omega)} \leq C \|\pi^m\|_{L_\varkappa(\Omega)}^{\varkappa-1}. \end{cases}$$

From the latter estimate it follows that  $\eta \in L_{\varkappa'}(-1, -\delta; \mathring{W}_{\varkappa'}^1(\Omega)) \subset L_{l'}(-1, -\delta; \mathring{W}_{s'}^1(\Omega))$ . Substituting this  $\eta$  into identity (2.20), we obtain  $\pi^m = 0$ . This implies  $q^m = \tilde{q}^m + \text{const}$  and we obtain the inclusion  $q^m \in W_{s, l}^{1,0}(\Omega \times ]-1, -\delta[)$ . Moreover, from (2.18), we find

$$\begin{aligned} & \|v^m\|_{W_{s, l}^{2,1}(B^+(\frac{1}{2}) \times ]-\frac{1}{4}, -\delta[)} + \|\nabla q^m\|_{L_{s, l}(B^+(\frac{1}{2}) \times ]-\frac{1}{4}, -\delta[)} \leq \\ & \leq C \left( \|\tilde{f}^m\|_{L_{s, l}(Q^+(\frac{2}{3}))} + \|v^m\|_{W_{s, l}^{1,0}(Q^+(\frac{2}{3}))} + \|q^m - b\|_{L_{s, l}(Q^+(\frac{2}{3}))} \right) \end{aligned}$$

where  $C$  is independent on  $m$  and  $\delta$ . Using identities  $v^m = \zeta u^m$ ,  $q^m = \zeta p^m$ ,  $\zeta \equiv 1$  on  $Q^+(\frac{2}{3})$  and expression (2.9) for  $\tilde{f}^m$ , we arrive at the estimate

$$\begin{aligned} & \|u^m\|_{W_{s, l}^{2,1}(B^+(\frac{1}{2}) \times ]-\frac{1}{4}, -\delta[)} + \|\nabla p^m\|_{L_{s, l}(B^+(\frac{1}{2}) \times ]-\frac{1}{4}, -\delta[)} \leq \\ & \leq C \left( \|f^m\|_{L_{s, l}(Q^+(\frac{2}{3}))} + \|u^m\|_{W_{s, l}^{1,0}(Q^+(\frac{2}{3}))} + \|p^m\|_{L_{s, l}(Q^+(\frac{2}{3}))} \right). \end{aligned}$$

Making use of (2.14) we conclude that

$$u \in W_{s,l}^{2,1} \left( B^+(\tfrac{1}{2}) \times ] - \tfrac{1}{4}, -\delta[ \right), \quad p \in W_{s,l}^{1,0} \left( B^+(\tfrac{1}{2}) \times ] - \tfrac{1}{4}, -\delta[ \right),$$

and the estimate

$$\begin{aligned} & \|u\|_{W_{s,l}^{2,1}(B^+(\frac{1}{2}) \times ]-\frac{1}{4}, -\delta[)} + \|\nabla p\|_{L_{s,l}(B^+(\frac{1}{2}) \times ]-\frac{1}{4}, -\delta[)} \leq \\ & \leq C \left( \|f\|_{L_{s,l}(Q^+(\frac{2}{3}))} + \|u\|_{W_{s,l}^{1,0}(Q^+(\frac{2}{3}))} + \|p\|_{L_{s,l}(Q^+(\frac{2}{3}))} \right) \end{aligned}$$

holds for any  $\delta \in ]0, \frac{1}{12}[$  with  $C$  independent on  $\delta$ . The last inequality provides the required properties of  $u$  and  $p$ . Theorem 2.5 is proved.  $\square$

Now, we are able to prove Theorem 2.3:

**Proof of Theorem 2.3.** For any  $k = 0, 1, \dots$  denote  $s_k = \frac{ns}{n-ks}$  if  $n > ks$  and  $\frac{ns}{n-ks} < m$  and  $s_k = m$  otherwise. Denote also  $N = \min\{k \in \mathbb{N} : s_k = m\}$  and  $\rho_k = \frac{1}{2} + \frac{1}{2^{k+1}}$ .

Using obvious modification of Theorem 2.2 and Theorem 2.5, we observe that if  $u \in W_{s_k,l}^{1,0}(Q^+(\rho_k))$  and  $p \in L_{s_k,l}(Q^+(\rho_k))$  is a weak solution of problem (2.2) in  $Q^+(\rho_k)$ , then  $u \in W_{s_{k+1},l}^{2,1}(Q^+(\rho_{k+1}))$  and  $p \in W_{s_{k+1},l}^{1,0}(Q^+(\rho_{k+1}))$  and the following estimate holds:

$$\begin{aligned} & \|u\|_{W_{s_{k+1},l}^{2,1}(Q^+(\rho_{k+1}))} + \|\nabla p\|_{L_{s_{k+1},l}(Q^+(\rho_{k+1}))} \leq \\ & \leq C \left( \|f\|_{L_{m,l}(Q^+)} + \|u\|_{W_{s_k,l}^{1,0}(Q^+(\rho_k))} + \|p\|_{L_{s_k,l}(Q^+(\rho_k))} \right). \end{aligned} \quad (2.21)$$

Moreover, due to the imbedding  $W_{s_k}^1(B^+(\rho_{k+1})) \hookrightarrow L_{s_{k+1}}(B^+(\rho_{k+1}))$ , we find the estimate

$$\begin{aligned} & \|u\|_{W_{s_{k+1},l}^{1,0}(Q^+(\rho_{k+1}))} + \|p\|_{L_{s_{k+1},l}(Q^+(\rho_{k+1}))} \leq \\ & \leq C \left( \|u\|_{W_{s_k,l}^{2,1}(Q^+(\rho_{k+1}))} + \|p\|_{W_{s_k,l}^{1,0}(Q^+(\rho_{k+1}))} \right). \end{aligned} \quad (2.22)$$

Iterating (2.21) and (2.22) from  $k = 0$  to  $k = N$  we finally obtain the bound

$$\begin{aligned} & \|u\|_{W_{s_N,l}^{2,1}(Q^+(\frac{1}{2}))} + \|\nabla p\|_{L_{s_N,l}(Q^+(\frac{1}{2}))} \leq \\ & \leq C^N \left( \|f\|_{L_{m,l}(Q^+)} + \|u\|_{W_{s_0,l}^{1,0}(Q^+)} + \|p\|_{L_{s_0,l}(Q^+)} \right). \end{aligned}$$

This estimate is equivalent to (2.6). Theorem 2.3 is proved.  $\square$

**Proof of Theorem 2.4.** Theorem 2.4 follows from Theorem 2.3 and the following imbedding theorem for anisotropic Sobolev spaces (see [1]):

$$\begin{aligned} & W_{m,l}^{2,1}(Q^+(\tfrac{1}{2})) \hookrightarrow C^{\beta, \frac{\beta}{2}}(\bar{Q}^+(\tfrac{1}{2})), \quad \text{if } m > \frac{3l}{2(l-1)} \quad \text{and} \quad \beta = 2 - \frac{3}{m} - \frac{2}{l} \\ & \|u\|_{C^{\beta, \frac{\beta}{2}}(\bar{Q}^+(\frac{1}{2}))} \leq C \|u\|_{W_{m,l}^{2,1}(Q^+(\frac{1}{2}))}, \quad \forall u \in W_{m,l}^{2,1}(Q^+(\tfrac{1}{2})). \end{aligned}$$

Theorem 2.4 is proved.  $\square$

### 3 Proof of the basic $\varepsilon$ -regularity condition

In this section we consider the Navier-Stokes system in a half-cylinder  $Q^+$

$$\begin{cases} \partial_t u + (u \cdot \nabla)u - \Delta u + \nabla p = 0 \\ \operatorname{div} u = 0 \\ u|_{x_3=0} = 0 \end{cases} \quad \text{in } Q^+ \quad (3.1)$$

The aim of this section is to provide a proof of the following theorem which is the boundary analogue of  $\varepsilon$ -regularity condition (1.3):

**Theorem 3.1** *For any given  $\alpha \in ]0, \frac{2}{3}[$  there exists a constant  $\varepsilon_* > 0$  depending only on  $\alpha$  such that for any boundary suitable weak solution  $u$  and  $p$  in  $Q^+$  subject to the condition*

$$\int_{Q^+} \left( |u|^3 + |p|^{\frac{3}{2}} \right) dxdt < \varepsilon_*, \quad (3.2)$$

*the velocity field  $u \in C^{\alpha, \frac{\alpha}{2}}(\bar{Q}^+(\frac{1}{2}))$ .*

We prove Theorem 3.1 following the method developed in [24]. This method is based on the indirect approach in the regularity theory (see terminology, for example, in [9]) and its crucial step is the decay estimate of Theorem 3.2. For a direct proof of partial regularity in the Navier-Stokes theory, we refer to [25].

**Theorem 3.2** *For any  $\theta \in ]0, \frac{1}{2}[$ ,  $\beta \in ]0, \frac{2}{3}[$ , there exists a constant  $\varepsilon_0(\theta, \beta) > 0$  such that, for any boundary suitable weak solution  $u$  and  $p$  to system (3.1) in  $Q^+$ , the following implication holds:*

$$\text{if } Y_1(u, p) < \varepsilon_0 \quad \text{then} \quad Y_\theta(u, p) \leq C_* \theta^\beta Y_1(u, p).$$

Here  $C_* > 0$  is some absolute constant.

Here we denote

$$Y_\theta(u, p) := \left( \int_{Q^+(\theta)} |u|^3 dxdt \right)^{1/3} + \theta \left( \int_{Q^+(\theta)} |p - [p]_{B^+(\theta)}|^{3/2} dxdt \right)^{2/3},$$

where for any  $f \in L_1(Q^+(\theta))$  we denote

$$\int_{Q^+(\theta)} f(x, t) dxdt = \frac{1}{|Q^+(\theta)|} \int_{Q^+(\theta)} f(x, t) dxdt, \quad [f]_{B^+(\theta)} = \frac{1}{|B^+(\theta)|} \int_{B^+(\theta)} f(x, t) dx$$

**Proof of Theorem 3.2:** Arguing by contradiction, we assume there is a number  $\theta \in ]0, \frac{1}{2}[$ , the sequence  $\varepsilon_h \rightarrow 0$  and functions  $u^h$  and  $p^h$  which are the boundary suitable weak solutions in the sense of Definition 1.2 satisfying relations

$$Y_1(u^h, p^h) = \varepsilon_h \rightarrow 0, \quad Y_\theta(u^h, p^h) \geq C_* \theta^\beta \varepsilon_h.$$

We introduce new functions

$$v^h = \frac{1}{\varepsilon_h} u^h, \quad q^h = \frac{1}{\varepsilon_h} (p^h - [p^h]_{B^+}).$$

They meet relations

$$Y_1(v^h, q^h) = 1, \quad Y_\theta(v^h, q^h) \geq C_* \theta^\beta, \quad (3.3)$$

as well as the system

$$\begin{cases} \partial_t v^h + \varepsilon_h \operatorname{div}(v^h \otimes v^h) - \Delta v^h + \nabla q^h = 0 \\ \operatorname{div} v^h = 0 \\ v^h|_{x_3=0} = 0, \end{cases} \quad \text{in } Q^+, \quad (3.4)$$

which holds in the sense of distributions and the boundary condition is understood in the sense of traces. Moreover, functions  $v^h$  and  $q^h$  satisfy the local energy inequality

$$\begin{aligned} & \int_{B^+} \zeta(x, t) |v^h(x, t)|^2 dx + 2 \int_{-1}^t \int_{B^+} \zeta |\nabla v^h|^2 dx dt \leq \\ & \leq \int_{-1}^t \int_{B^+} \left\{ |v^h|^2 (\partial_t \zeta + \Delta \zeta) + v^h \cdot \nabla \zeta (\varepsilon_h |v^h|^2 + 2q^h) \right\} dx dt \end{aligned} \quad (3.5)$$

for a.e.  $t \in ]-1, 0[$  and all nonnegative  $\zeta \in C^\infty(\bar{Q})$  vanishing near  $\partial' Q$ .

From (3.3), we derive the estimate

$$\|v^h\|_{L_3(Q^+)} + \|q^h\|_{L_{\frac{3}{2}}(Q^+)} \leq C. \quad (3.6)$$

Picking up a cut-off function  $\zeta$  so that  $\zeta \equiv 1$  on  $Q^+(\frac{3}{4})$  and taking into account (3.6), we find

$$\|v^h\|_{L_{2,\infty}(Q^+(\frac{3}{4}))} + \|v^h\|_{W_2^{1,0}(Q^+(\frac{3}{4}))} \leq C. \quad (3.7)$$

The known multiplicative inequality allows us to conclude that

$$\|v^h\|_{L_{\frac{10}{3}}(Q^+(\frac{3}{4}))} \leq C. \quad (3.8)$$

Another bound easily follows from (3.4) and has the form

$$\|\partial_t v^h\|_{L_{\frac{3}{2}}((-\frac{3}{4})^2, 0; W_{\frac{3}{2}}^{-1}(B^+(\frac{3}{4})))} \leq C. \quad (3.9)$$

Estimates (3.6), (3.7) provide the existence of subsequences  $\{v^h\}$  and  $\{q^h\}$  with the following properties

$$v^h \rightharpoonup v^0 \quad \text{in } W_2^{1,0}(Q^+(\frac{3}{4})) \cap L_3(Q^+), \quad (3.10)$$

$$q^h \rightharpoonup q^0 \quad \text{in } L_{\frac{3}{2}}(Q^+). \quad (3.11)$$

Routine compactness arguments imply

$$v^h \rightarrow v^0 \quad \text{in } L_3(Q^+(\frac{3}{4})), \quad (3.12)$$



Convergence (3.10) – (3.11) allow us to pass to the limit in equations (3.4) (if we take these equations in the weak form). So,  $v^0$  and  $q^0$  is a weak solution to the system

$$\begin{cases} \partial_t v^0 - \Delta v^0 + \nabla q^0 = 0 \\ \operatorname{div} v^0 = 0 \\ v^0|_{x_3=0} = 0. \end{cases} \quad \text{in } Q^+(\tfrac{3}{4}), \quad (3.13)$$

Moreover, from the second relation in (3.3), we deduce the estimate

$$\liminf_{h \rightarrow \infty} Y_\theta(v^h, q^h) \geq C_* \theta^\beta. \quad (3.14)$$

On the other hand, below we will show that

$$\limsup_{h \rightarrow \infty} Y_\theta(v^h, q^h) \leq C_{**} \theta^\beta, \quad (3.15)$$

with a constant  $C_{**} > 0$ . Taking in (3.14) a constant  $C_* > C_{**}$  we arrive at a contradiction between (3.14) and (3.15). This will complete our proof Theorem 3.2.

To prove (3.15), we split  $Y_\theta(v^h, q^h)$  onto two parts:

$$Y_\theta(v^h, q^h) = Y_\theta^1(v^h) + Y_\theta^2(q^h),$$

where

$$Y_\theta^1(v^h) \equiv \left( \int_{Q^+(\theta)} |v^h|^3 dx dt \right)^{\frac{1}{3}}, \quad Y_\theta^2(q^h) \equiv \theta \left( \int_{Q^+(\theta)} |q^h|^{\frac{3}{2}} dx dt \right)^{\frac{2}{3}}$$

As  $\theta \in ]0, \frac{1}{2}[$ , strong convergence (3.12) gives us the following

$$\lim_{h \rightarrow \infty} Y_\theta^1(v^h) = Y_\theta^1(v^0). \quad (3.16)$$

Since  $v^0 \in W_2^{1,0}(Q^+(\frac{3}{4}))$ ,  $q^0 \in L_{\frac{3}{2}}(Q^+(\frac{3}{4}))$  are a weak solution to the Stokes system (3.13) in  $Q^+(\frac{3}{4})$  (in the sense of Definition 2.1), one can apply Theorems 2.5 and 2.4 find that  $v^0 \in C^{\beta, \frac{\beta}{2}}(\bar{Q}^+(\frac{1}{2}))$  with the estimate

$$\|v^0\|_{C^{\beta, \frac{\beta}{2}}(\bar{Q}^+(\frac{1}{2}))} \leq C \left( \|\nabla v^0\|_{L_{\frac{3}{2}}(Q^+(\frac{3}{4}))} + \|q^0\|_{L_{\frac{3}{2}}(Q^+)} \right). \quad (3.17)$$

Thanks to estimates (3.6), (3.7) and the lower semicontinuity of the corresponding norms with respect to weak convergence (3.10), (3.11), the right-hand side of (3.17) can be estimated by some absolute constant  $C$ . As  $v^0|_{x_3=0} = 0$ , relation (3.17) yeilds the estimate

$$Y_\theta^1(v^0) \leq C \theta^\beta \|v^0\|_{C^{\beta, \frac{\beta}{2}}(\bar{Q}^+(\frac{1}{2}))} \leq C_0 \theta^\beta. \quad (3.18)$$

Combining (3.16) and (3.18), we find the estimate

$$\limsup_{h \rightarrow \infty} Y_\theta(v^h, q^h) \leq C_0 \theta^\beta + \limsup_{h \rightarrow \infty} Y_\theta^2(q^h) \quad (3.19)$$

So, to show (3.15), we need to estimate the second term in the right-hand side of (3.19). For this purpose, let us define

$$f^h = -\varepsilon_h \operatorname{div}(v^h \otimes v^h) \quad \text{in } Q^+(\tfrac{3}{4}).$$

From Hölder inequality, it follows that  $f^h \in L_{\frac{9}{8}, \frac{3}{2}}(Q^+(\frac{3}{4}))$  and using (3.7) we obtain

$$\|f^h\|_{L_{\frac{9}{8}, \frac{3}{2}}(Q^+(\frac{3}{4}))} \leq C \varepsilon_h \|v^h\|_{L_{2, \infty}(Q^+(\frac{3}{4}))}^{\frac{2}{3}} \|\nabla v^h\|_{L_2(Q^+(\frac{3}{4}))}^{\frac{4}{3}} \rightarrow 0 \quad \text{as } h \rightarrow \infty. \quad (3.20)$$

Assume  $\Omega \subset \mathbb{R}^3$  is a smooth canonical domain such that  $B^+(\frac{5}{8}) \subset \Omega \subset B^+(\frac{3}{4})$  and denote  $\tilde{Q} := \Omega \times ]-\frac{9}{16}, 0[$ . It is known, see, for example, [39], that there exist a unique pair of functions  $\hat{v}^h \in W_{\frac{9}{8}, \frac{3}{2}}^{2,1}(\tilde{Q})$ ,  $q_1^h \in W_{\frac{9}{8}, \frac{3}{2}}^{1,0}(\tilde{Q})$ ,  $[q_1^h]_{\Omega} = 0$  a.e.  $t \in ]-\frac{9}{16}, 0[$ , which obey the following initial-boundary value problem

$$\begin{cases} \partial_t \hat{v}^h - \Delta \hat{v}^h + \nabla q_1^h = f^h & \text{in } \tilde{Q}, \\ \operatorname{div} \hat{v}^h = 0 \\ \hat{v}^h|_{t=-\frac{9}{16}} = 0, \quad \hat{v}^h|_{\partial\Omega \times ]-\frac{9}{16}, 0[} = 0, \end{cases}$$

and is subject to the estimate

$$\|\hat{v}^h\|_{W_{\frac{9}{8}, \frac{3}{2}}^{2,1}(\tilde{Q})} + \|q_1^h\|_{W_{\frac{9}{8}, \frac{3}{2}}^{1,0}(\tilde{Q})} \leq C \|f^h\|_{L_{\frac{9}{8}, \frac{3}{2}}(Q^+(\frac{3}{4}))}. \quad (3.21)$$

From (3.20), (3.21), and from the imbedding  $W_{\frac{9}{8}}^1(\Omega) \hookrightarrow L_{\frac{3}{2}}(\Omega)$ , we can conclude

$$Y_{\theta}^2(q_1^h) \leq C \theta^{-2} \|\nabla q_1^h\|_{L_{\frac{9}{8}, \frac{3}{2}}(\tilde{Q})} \leq C \theta^{-2} \|f^h\|_{L_{\frac{9}{8}, \frac{3}{2}}(Q^+(\frac{3}{4}))} \rightarrow 0 \quad \text{as } h \rightarrow \infty. \quad (3.22)$$

Now, consider the functions  $\tilde{v}^h \equiv v^h - \hat{v}^h$ ,  $q_2^h = q^h - q_1^h$ . Note that  $\tilde{v}^h \in W_{\frac{9}{8}, \frac{3}{2}}^{1,0}(\tilde{Q})$ ,  $q_2^h \in L_{\frac{9}{8}, \frac{3}{2}}(\tilde{Q})$  and, hence,  $\tilde{v}^h$  and  $q_2^h$  is a weak solution (in the sense of Definition 2.1) of the homogeneous Stokes system in  $\tilde{Q}$

$$\begin{cases} \partial_t \tilde{v}^h - \Delta \tilde{v}^h + \nabla q_2^h = 0 & \text{in } \tilde{Q}, \\ \operatorname{div} \tilde{v}^h = 0 \\ \tilde{v}^h|_{x_3=0} = 0. \end{cases}$$

Let us take  $m := \frac{9}{2-3\beta}$ ,  $m \in ]\frac{9}{2}, +\infty[$ . By an obvious modification of Theorem 2.3 with  $f \equiv 0$  we obtain inclusions  $\tilde{v}^h \in W_{m, \frac{3}{2}}^{2,1}(Q^+(\frac{1}{2}))$ ,  $q_2^h \in W_{m, \frac{3}{2}}^{2,1}(Q^+(\frac{1}{2}))$  and the estimate

$$\|\nabla q_2^h\|_{L_{m, \frac{3}{2}}(Q^+(\frac{1}{2}))} \leq C \left( \|\tilde{v}^h\|_{W_{\frac{9}{8}, \frac{3}{2}}^{1,0}(\tilde{Q})} + \|q_2^h\|_{L_{\frac{9}{8}, \frac{3}{2}}(\tilde{Q})} \right).$$

The right-hand side of the last inequality can be controlled by majorants which do not depend on  $h$ . Indeed, as  $\tilde{v}^h = v^h - \hat{v}^h$ , from (3.7), (3.21), we find

$$\|\nabla \tilde{v}^h\|_{L_{\frac{9}{8}, \frac{3}{2}}(Q^+(\frac{3}{4}))} \leq \|\nabla v^h\|_{L_2(Q^+(\frac{3}{4}))} + \|\nabla \hat{v}^h\|_{L_{\frac{9}{8}, \frac{3}{2}}(Q^+(\frac{3}{4}))} \leq C.$$

As  $q_2^h = q^h - q_1^h$ , using (3.6), (3.20), (3.21) we get

$$\|q_2^h\|_{L_{\frac{9}{8}, \frac{3}{2}}(\tilde{Q})} \leq C \left( \|q^h\|_{L_{\frac{3}{2}}(Q^+)} + \|q_1^h\|_{L_{\frac{9}{8}, \frac{3}{2}}(\tilde{Q})} \right) \leq C \left( 1 + \|f^h\|_{L_{\frac{9}{8}, \frac{3}{2}}(Q^+(\frac{3}{4}))} \right) \leq C.$$

Hence, it has been shown that

$$\|\nabla q_2^h\|_{L_{m, \frac{3}{2}}(Q^+(\frac{1}{2}))} \leq C. \quad (3.23)$$

Using (3.23) and the Poincaré and Hölder inequalities, we find

$$Y_\theta^2(q_2^h) \leq C\theta^\beta \|\nabla q_2^h\|_{L_{m, \frac{3}{2}}(Q^+(\frac{1}{2}))} \leq C_1\theta^\beta. \quad (3.24)$$

Combining (3.24) with (3.22), we show

$$\limsup_{h \rightarrow \infty} Y_\theta^2(q^h) \leq \limsup_{h \rightarrow \infty} Y_\theta^2(q_1^h) + \limsup_{h \rightarrow \infty} Y_\theta^2(q_2^h) \leq 0 + C_1\theta^\beta = C_1\theta^\beta.$$

Hence from (3.19) we obtain (3.15) with  $C_{**} = C_0 + C_1$  which contradicts to (3.14) if we take  $C_* > C_{**}$ . Theorem 3.2 is proved.  $\square$

**Theorem 3.3** *Let  $C_* > 1$  be the absolute constant defined by Theorem 3.2. Assume  $\beta \in ]0, \frac{2}{3}[$ ,  $\theta \in ]0, \frac{1}{2}[$  are arbitrary. Denote by  $\varepsilon_0 > 0$ ,  $\varepsilon_0 = \varepsilon_0(\theta, \beta)$  a constant determined by Theorem 3.2. Then, for any boundary suitable weak solution  $u$  and  $p$  to the Navier-Stokes system in  $Q^+$  and for any  $k = 0, 1, 2, \dots$ , the following is true:*

$$\text{if } Y_{\theta^k}(u, p) < \varepsilon_0 \quad \text{then} \quad Y_{\theta^{k+1}}(u, p) \leq C_*\theta^\beta Y_{\theta^k}(u, p) \quad (3.25)$$

**Proof:** Define functions  $u^{\theta^k}$  and  $p^{\theta^k}$  by formulas

$$\begin{aligned} u^{\theta^k}(x, t) &:= \theta^k u(\theta^k x, \theta^{2k} t) \\ p^{\theta^k}(x, t) &:= \theta^{2k} p(\theta^k x, \theta^{2k} t) \end{aligned} \quad (x, t) \in Q^+.$$

As  $u$  and  $p$  is a boundary suitable weak solution of the Navier-Stokes equations in  $Q^+(\theta^k)$ , the functions  $u^{\theta^k}$  and  $p^{\theta^k}$  are a boundary suitable weak solution of the Navier-Stokes system in  $Q^+$ . Moreover,

$$Y_1(u^{\theta^k}, p^{\theta^k}) = Y_{\theta^k}(u, p) < \varepsilon_0,$$

and, hence, by Theorem 3.2

$$Y_\theta(u^{\theta^k}, p^{\theta^k}) \leq C_*\theta^\alpha Y_1(u^{\theta^k}, p^{\theta^k}).$$

From this inequality, the conclusion of the implication (3.25) follows by change of variables. Theorem 3.3 is proved.  $\square$

**Theorem 3.4** *Let  $C_* > 1$  be the absolute constant defined by Theorem 3.2 and let  $\alpha \in ]0, \frac{2}{3}[$ ,  $\beta \in ]\alpha, \frac{2}{3}[$  be arbitrary. Assume a number  $\theta \in ]0, \frac{1}{2}[$  is fixed in such a way that*

$$C_*\theta^{\beta-\alpha} < 1, \quad (3.26)$$

*and let  $\varepsilon > 0$ ,  $\varepsilon_0 = \varepsilon_0(\theta, \beta)$  be a constant determined by Theorem 3.2. Then, for any boundary suitable weak solution  $u$  and  $p$  of the Navier-Stokes system in  $Q^+$  and for any  $k = 0, 1, 2, \dots$ , the following is valid:*

$$\text{if } Y_1(u, p) < \varepsilon_0 \quad \text{then for any } k = 0, 1, 2, \dots \quad \left\{ \begin{array}{l} Y_{\theta^k}(u, p) < \varepsilon_0 \\ Y_{\theta^{k+1}}(u, p) < \theta^{\alpha(k+1)} Y_1(u, p) \end{array} \right.$$

**Proof:** The proof follows easily from Theorem 3.3 by induction in  $k$ .  $\square$

**Theorem 3.5** Assume  $\alpha \in ]0, \frac{2}{3}[$  is arbitrary and take  $\beta = \frac{1}{2}(\alpha + \frac{2}{3})$ . Let us fix  $\theta \in ]0, \frac{1}{2}[$ ,  $\theta = \theta(\alpha)$  so that (3.26) holds and let  $\varepsilon_0 > 0$ ,  $\varepsilon_0 = \varepsilon_0(\theta, \beta)$  be a constant determined by Theorem 3.2. Denote  $\varepsilon'_* := \varepsilon_0(\theta, \beta)$  and note that  $\varepsilon'_* > 0$  actually depends only on  $\alpha$ . Then for any  $z_0 = (x_0, t_0)$ ,  $x_0 \in \partial\mathbb{R}_+^3$ , and for any boundary suitable weak solution  $u$  and  $p$  of the Navier-Stokes equations in  $Q^+(z_0, R)$ , the following is valid:

$$\begin{aligned} & \text{if} \quad R Y_{z_0, R}(u, p) < \varepsilon'_* \\ & \text{then for any} \quad 0 < r < R \quad Y_{z_0, r}(u, p) \leq C(\alpha) \left(\frac{r}{R}\right)^\alpha Y_{z_0, R}(u, p). \end{aligned}$$

Here,

$$Y_{z_0, R}(u, p) := \left( \int_{Q^+(z_0, R)} |u|^3 dxdt \right)^{\frac{1}{3}} + R \left( \int_{Q^+(z_0, R)} |p - [p]_{B^+(x_0, R)}|^{\frac{3}{2}} dxdt \right)^{\frac{2}{3}}$$

**Proof:** The proof follows easily from Theorem 3.4 by scaling transformation (1.1), if we fix  $k \in \mathbb{N} \cup \{0\}$  in such a way that  $\theta^{k+1}R \leq r < \theta^k R$ .  $\square$

**Proof of Theorem 3.1:** Let

$$\begin{aligned} \bar{Y}_{z_0, R}(u, p) &:= \left( \int_{Q(z_0, R) \cap Q^+} |u - (u)_{Q(z_0, R) \cap Q^+}|^3 dxdt \right)^{\frac{1}{3}} + \\ &+ R \left( \int_{Q(z_0, R) \cap Q^+} |p - [p]_{B(x_0, R) \cap B^+}|^{\frac{3}{2}} dxdt \right)^{\frac{2}{3}}, \end{aligned}$$

where  $(u)_{Q(z_0, R) \cap Q^+}$  denotes the space-time average of  $u$  over the set  $Q(z_0, R) \cap Q^+$ , and  $[p]_{B(x_0, R) \cap B^+}$  denotes the spatial average of  $p$  over the set  $B(x_0, R) \cap B^+$ .

For any  $x_0 \in \mathbb{R}_+^3$ ,  $x_0 = (x_1^0, x_2^0, x_3^0)$ ,  $z_0 = (x_0, t_0)$ , we denote by  $x'_0$  the point with coordinates  $(x_1^0, x_2^0, 0)$ ,  $z'_0 := (x'_0, t_0)$ , and  $d(x_0) := \text{dist}\{x_0, \partial\mathbb{R}_+^3\} = x_3^0$ .

From the internal regularity theory (see, for example, [28]) we know there is a constant  $\varepsilon''_* > 0$  depending only on  $\alpha$  such that for any  $z_0 \in Q^+$ ,  $R > 0$  the following implication holds:

$$Q(z_0, R) \subset Q^+, \quad R \bar{Y}_{z_0, R}(u, p) < \varepsilon''_* \implies \bar{Y}_{z_0, r}(u, p) \leq c \left(\frac{r}{R}\right)^\alpha \bar{Y}_{z_0, R}(u, p) \quad (3.27)$$

On the other hand, Theorem 3.5 reads that there is a constant  $\varepsilon'_* > 0$  depending only on  $\alpha$  such that for any  $z'_0 \in \partial\mathbb{R}_+^3$ ,  $R > 0$ , we have:

$$Q^+(z'_0, R) \subset Q^+, \quad R Y_{z'_0, R}(u, p) < \varepsilon'_* \implies Y_{z'_0, r}(u, p) \leq c \left(\frac{r}{R}\right)^\alpha Y_{z'_0, R}(u, p) \quad (3.28)$$

Besides, it is easy to see that there is an absolute constant  $c_1$  such that for any  $z_0 \in B^+(\frac{1}{2})$ ,  $z'_0 \in \bar{B}^+(\frac{1}{2}) \cap \partial\mathbb{R}_+^3$

$$\bar{Y}_{z_0, \frac{1}{4}}(u, p) \leq c_1 Y_1(u, p), \quad Y_{z'_0, \frac{1}{4}}(u, p) \leq c_1 Y_1(u, p) \quad (3.29)$$

Assume  $z_0 = (x_0, t_0) \in \bar{Q}^+(\frac{1}{2})$  is arbitrary and  $0 < r < \frac{1}{8}$ . Denote  $d := d(x_0)$ . There are three possible cases:

Case 1:  $0 \leq d < r < \frac{1}{8}$

Case 2:  $0 < r \leq d < \frac{1}{8}$

Case 3:  $\frac{1}{8} \leq d \leq \frac{1}{2}$

In Case 1 we have  $\bar{Y}_{z_0,r}(u,p) \leq 2 Y_{z'_0,2r}(u,p)$ . Let us fix  $\varepsilon_1 := \frac{\varepsilon'_*}{c_1}$  where  $c_1$  is fixed in (3.29). Then if we assume  $Y_1(u,p) < \varepsilon_1$  from (3.29) we obtain  $Y_{z'_0,\frac{1}{4}}(u,p) < \varepsilon'_*$  and hence with the help of (3.28) we obtain

$$\bar{Y}_{z_0,r}(u,p) \leq 2 Y_{z'_0,2r}(u,p) \leq 2 c \left( \frac{2r}{\frac{1}{4}} \right)^\alpha Y_{z'_0,\frac{1}{4}}(u,p) \leq c \varepsilon'_* r^\alpha.$$

In Case 2 we have  $\bar{Y}_{z_0,d}(u,p) \leq 2 Y_{z'_0,2d}(u,p)$ . If assume  $Y_1(u,p) < \varepsilon_1$ , (3.29) yields  $Y_{z'_0,\frac{1}{4}}(u,p) < \varepsilon'_*$  and hence with the help of (3.28), (3.29) we state that:

$$\bar{Y}_{z_0,d}(u,p) \leq 2 Y_{z'_0,2d}(u,p) \leq 2 c \left( \frac{2d}{\frac{1}{4}} \right)^\alpha Y_{z'_0,\frac{1}{4}}(u,p) \leq c_2 Y_1(u,p) d^\alpha.$$

Taking into account that  $d < \frac{1}{8}$ , we can conclude that  $d \bar{Y}_{z_0,d}(u,p) \leq c_2 Y_1(u,p)$ . Hence, if we fix  $\varepsilon_2 := \min\{\frac{\varepsilon'_*}{c_1}, \frac{\varepsilon''_*}{c_2}\}$  and assume  $Y_1(u,p) < \varepsilon_2$ , we can apply (3.27) with  $R = d$  and show

$$\bar{Y}_{z_0,r}(u,p) \leq c \left( \frac{r}{d} \right)^\alpha \bar{Y}_{z_0,d}(u,p) \leq c \varepsilon''_* r^\alpha$$

In Case 3 we have  $d \bar{Y}_{z_0,d}(u,p) \leq c_3 Y_1(u,p)$  with an absolute constant  $c_3 > 0$ . Hence if we take  $\varepsilon_3 := \frac{\varepsilon''_*}{c_3}$  and assume  $Y_1(u,p) < \varepsilon_3$  we observe that  $d \bar{Y}_{z_0,d}(u,p) < \varepsilon''_*$  and hence we can apply (3.27) with  $R = d$ . Taking into account  $\frac{1}{8} \leq d \leq \frac{1}{2}$ , we find

$$\bar{Y}_{z_0,r}(u,p) \leq c \left( \frac{r}{d} \right)^\alpha \bar{Y}_{z_0,d}(u,p) \leq c \varepsilon''_* r^\alpha.$$

Finally, if we fix  $\varepsilon_* := \min\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$  and assume  $Y_1(u,p) < \varepsilon_*$  then in all cases we get the estimate

$$\forall z_0 \in \bar{Q}^+(\frac{1}{2}), \quad \forall r \in (0, \frac{1}{8}) \quad \bar{Y}_{z_0,r}(u,p) \leq K r^\alpha,$$

with some  $K > 0$  depending only on  $\alpha$ ,  $\varepsilon'_*$  and  $\varepsilon''_*$ . From this estimate, we deduce Hölder continuity of  $u$  in the set  $\bar{Q}^+(\frac{1}{2})$  via Campanato criterion, see, for example, [3]. Theorem 3.1 is proved.  $\square$

## 4 Further results

In this section we discuss further results of  $\varepsilon$ -regularity theory for the Navier-Stokes system. We start with the Navier-Stokes equations in the neighborhood of a point  $x_0$  belonging to the smooth curvilinear part of the boundary  $\partial\Omega$  of a domain  $\Omega \subset \mathbb{R}^3$ . Namely, assume  $x_0 \in \partial\Omega$ , denote by  $\Omega(x_0, R)$  the intersection of some neighborhood of  $x_0$  with  $\Omega$  and consider the system

$$\begin{cases} \partial_t u - \Delta u + (u \cdot \nabla)u + \nabla p = f \\ \operatorname{div} u = 0 \\ u|_{\partial\Omega \times ]-R^2, 0[} = 0 \end{cases} \quad \text{in } \Omega(x_0, R) \times ]-R^2, 0[. \quad (4.1)$$

Without loss of generality we can assume that our Cartesian coordinate system is chosen in such a way that  $x_0$  coincides with its origin (i.e.  $x_0 = 0$ ) and the set  $\Omega(x_0, R)$  is described by relations

$$\Omega(x_0, R) = \left\{ x = (x_1, x_2, x_3) \in \mathbb{R}^3 \mid x' \in S_R, \varphi(x') < x_3 < \varphi(x') + \sqrt{R^2 - |x'|^2} \right\}. \quad (4.2)$$

Here we denote  $x' := (x_1, x_2)$  and  $S_R := \{ x' \in \mathbb{R}^2 \mid \sqrt{x_1^2 + x_2^2} < R \}$ . With this assumptions the boundary condition in (4.1) is equivalent to the relation

$$u|_{x_3=\varphi(x')} = 0.$$

We assume  $\varphi$  is of class  $W_\infty^3$  (i.e. its second derivatives are Lipschitz continuous) and the Cartesian coordinate system is chosen in such a way that the following relations hold

$$\varphi(0) = 0, \quad \nabla\varphi(0) = 0, \quad \|\varphi\|_{W_\infty^3(S_R)} \leq \mu. \quad (4.3)$$

Now we apply the diffeomorphism flattening the boundary, or, in other words, we introduce new coordinates  $y = \psi(x)$  by formulas

$$\psi : \Omega_R \rightarrow B_R^+, \quad y = \psi(x) = \begin{pmatrix} x' \\ x_3 - \varphi(x') \end{pmatrix}, \quad (4.4)$$

$$x \in \Omega(x_0, R) \iff y \in B_R^+.$$

Denote

$$v := u \circ \psi^{-1}, \quad q := p \circ \psi^{-1}, \quad \tilde{f} := f \circ \psi^{-1}.$$

Then for  $y = \psi(x)$  we have relations

$$\nabla p(x) = \hat{\nabla}_\varphi q(y), \quad \Delta u(x) = \hat{\Delta}_\varphi v(y), \quad \operatorname{div} u(x) = (\hat{\nabla}_\varphi \cdot v)(y).$$

where  $\hat{\Delta}_\varphi$  and  $\hat{\nabla}_\varphi$  are the differential operators with variable coefficients defined via a function  $\varphi$  by formulas

$$\begin{aligned} \hat{\Delta}_\varphi v &:= \Delta v - 2v_{,\alpha 3}\varphi_{,\alpha} + v_{,33}|\nabla'\varphi|^2 - v_{,3}\Delta'\varphi, \\ \hat{\nabla}_\varphi \cdot v &:= \operatorname{div} v - v_{\alpha,3}\varphi_{,\alpha}, \\ \hat{\nabla}_\varphi q &:= \nabla q - q_{,3} \begin{pmatrix} \nabla'\varphi \\ 0 \end{pmatrix}. \end{aligned}$$

Here we assume summation from 1 to 2 over repeated Greek indexes and  $\nabla'$  and  $\Delta'$  denote the gradient and Laplacian with respect to  $(y_1, y_2)$  variables.

The Navier-Stokes system (4.1) in  $\Omega(x_0, R) \times (-R^2, 0)$  in  $x$ -variables is transformed to the following system with variable coefficients depending on the  $y$ -variables:

$$\begin{cases} \partial_t v - \hat{\Delta}_\varphi v + (v \cdot \hat{\nabla}_\varphi)v + \hat{\nabla}_\varphi q = \tilde{f} \\ \hat{\nabla}_\varphi \cdot v = 0 \\ v|_{y_3=0} = 0. \end{cases} \quad \text{in } Q^+(R), \quad (4.5)$$

We call this system *the Perturbed Navier-Stokes system*. Note that if the boundary  $\partial\Omega$  is smooth in the neighborhood of  $x_0$  then the coefficient of system (4.5) are also smooth.

The Perturbed Navier-Stokes system possesses the following scaling property: if functions  $v, q, f, \varphi$  satisfy (4.5) in the cylinder  $Q^+(R)$  with  $\varphi$  satisfying (4.3) then the functions

$$\begin{aligned} v^R(x, t) &= Rv(Rx, R^2t), & q^R(x, t) &= R^2q(Rx, R^2t), \\ f^R(x, t) &= R^3f(Rx, R^2t), & \varphi^R(x') &= \frac{1}{R}\varphi(Rx') \end{aligned} \quad (4.6)$$

satisfy the Perturbed Navier-Stokes system (4.5) in  $Q^+$  and from Taylor decomposition of the function  $\varphi^R$  one can obtain for  $R \leq 1$

$$\varphi^R(0) = 0, \quad \nabla' \varphi^R(0) = 0, \quad \|\varphi^R\|_{W_\infty^3(S_1)} \leq \mu R. \quad (4.7)$$

Hence, if assumptions (4.3) hold with some constant  $\mu$  whose value can be arbitrary large (in particular, this implies that the curvature of the boundary of  $\Omega$  in the neighborhood of  $x_0 \in \partial\Omega$  can be arbitrary) then applying diffeomorphism (4.4) and the scaling transformation (4.6) we can reduce the study of regularity of weak solutions to the Navier-Stokes system (4.1) in domain  $\Omega(x_0, R) \times ]-R^2, 0[$  to the study of the Perturbed Navier-Stokes system (4.5) in the canonical domain  $Q^+$ . Moreover, for any given  $\mu_* > 0$  choosing the radius  $R = R(\mu_*) > 0$  sufficiently small thanks to (4.7) we can assume that condition

$$\mu R \leq \mu_* \quad (4.8)$$

holds with some absolute constant  $\mu_* > 0$ . If the value  $\mu_*$  in (4.8) is chosen sufficiently small then variable coefficients in the Perturbed Navier-Stokes system (4.5) for functions  $v^R, q^R, f^R, \varphi^R$  in  $Q^+$  can be interpreted as small perturbations of the “constant coefficients” in the usual Navier-Stokes system (3.1) for functions  $v^R, q^R, f^R$  in  $Q^+$ .

The linear theory of strong solutions to the Stokes system extending results of [23] to the case of the curvilinear boundary was developed in [38]. Later, in [36] the similar theory was developed for the linear Perturbed Stokes system which is the linearization of (4.5). Moreover, in contrast to [38], in [36] the local estimates were obtained not for strong but for weak solutions. In particular, the analogs of our Theorems 2.2 — 2.5 were proved in [36] for the Perturbed Stokes system under the assumption that its coefficients are small perturbations of the constant coefficients of the usual Stokes system.

The linear theory developed in [38], [36] allows to prove the analogue of the basic  $\varepsilon$ -regularity condition (3.2) at the neighborhood of point  $x_0$  belonging to a curvilinear part of the boundary. To formulate a result we need to define boundary suitable weak solution to the Navier-Stokes equation near a curvilinear part of the boundary. The

definition of this class of solutions is analogous to one given in our Definition 1.2 for the case of flat boundaries, see details in [32]. So, we obtain the following result:

**Theorem 4.1** *Let  $\Omega(x_0, R)$  be defined by (4.2) where  $\varphi \in W_\infty^3(S_R)$  satisfies (4.3) and assume  $z_0 = (x_0, t_0)$ . There exist an absolute constant  $\varepsilon_* > 0$  and a constant  $R_* \in (0, R)$  depending only on  $\mu$  and  $R$  such that for any boundary suitable weak solution  $u$  and  $p$  of the Navier-Stokes equations (4.1) in  $\Omega(x_0, R) \times ]t_0 - R^2, t_0[$  the following is true: if there exists  $r \leq R_*$  such that*

$$\frac{1}{r^2} \int_{t_0 - r^2}^{t_0} \int_{\Omega(x_0, r)} \left( |u|^3 + |p|^{\frac{3}{2}} \right) dxdt < \varepsilon_* \quad (4.9)$$

*then  $u$  is Hölder continuous on  $\bar{\Omega}(x_0, \frac{r}{2}) \times [t_0 - \frac{r^2}{4}, t_0]$ .*

Theorem 4.1 is proved in [32]. Using this theorem one can obtain the following result that is a boundary version of the Caffarelli-Kohn-Nirenberg theorem, see [4]:

**Theorem 4.2** *Let all assumptions of Theorem 4.1 be satisfied. There exists an absolute constant  $\varepsilon_{**} > 0$  such that, for any boundary suitable weak solution  $u$  and  $p$  of the Navier-Stokes equations (4.1) in  $\Omega(x_0, R) \times ]t_0 - R^2, t_0[$ , the velocity field  $u$  is Hölder continuous in some neighborhood of  $z_0$  provided*

$$\limsup_{r \rightarrow 0} \frac{1}{r} \int_{t_0 - r^2}^{t_0} \int_{\Omega(x_0, r)} |\nabla u|^2 dxdt < \varepsilon_{**}. \quad (4.10)$$

Theorem 4.2 provides the following estimate of the parabolic Hausdorff measure of singular set:

**Theorem 4.3** *Let  $\Omega \subset \mathbb{R}^3$  be a domain whose boundary  $\partial\Omega$  is of class  $W_\infty^3$  and assume there is a constant  $\mu > 0$  such that for any  $x_0 \in \partial\Omega$  there exists a neighborhood  $\Omega(x_0, R)$  which can be described in an appropriate Cartesian coordinate system by formulas (4.2) with some function  $\varphi = \varphi_{x_0}$  satisfying conditions (4.3). Then for any boundary suitable weak solution  $u$  and  $p$  of the Navier-Stokes system in  $\Omega \times ]0, T[$  there exists a closed set  $\Sigma \subset \partial\Omega \times ]0, T[$  such that for any point  $z_0 \in (\partial\Omega \times ]0, T[) \setminus \Sigma$  the function  $u$  is Hölder continuous in some neighborhood of  $z_0$  and, moreover,*

$$\mathcal{P}^1(\Sigma) = 0,$$

*where  $\mathcal{P}^1(\Sigma)$  is the one-dimensional parabolic Hausdorff measure of  $\Sigma$ .*

It worthy to notice that there is an essential difference between  $\varepsilon$ -regularity conditions (4.9) and (4.10). Condition (4.10) requires smallness of functional

$$E(u, r) = \left( \frac{1}{r} \int_{t_0 - r^2}^{t_0} \int_{\Omega(x_0, r)} |\nabla u|^2 dydt \right)^{1/2}$$



uniformly with respect to all  $r \in (0, R_1)$  with some  $R_1 > 0$ , i.e.

$$\limsup_{r \rightarrow 0} E(u, r) < \varepsilon_0. \quad (4.11)$$

The question is whether (4.11) could be weakened to

$$\liminf_{r \rightarrow 0} E(u, r) < \varepsilon_0. \quad (4.12)$$

In general, an answer to this question is unknown. But if we assume *a priori* boundedness of scale invariant functionals of type (1.5) then it is possible to improve most of  $\varepsilon$ -regularity conditions, replacing in (4.11)  $\limsup$  with  $\liminf$ . Namely, the boundary suitable weak solution  $u$  and  $p$  to the Navier-Stokes system in  $Q^+$  is regular near the origin if (1.6) holds and besides one of the following conditions is valid:

- $\min \left\{ \liminf_{r \rightarrow 0} A(u, r), \liminf_{r \rightarrow 0} C(u, r), \liminf_{r \rightarrow 0} E(u, r), \liminf_{r \rightarrow 0} D(p, r) \right\} < \varepsilon_0$
- $\liminf_{r \rightarrow 0} C_2(u, r) < \varepsilon_0$ , where

$$C_2(u, r) := \left( \frac{1}{r^3} \int_{Q^+(r)} |u(x, t)|^2 \, dx dt \right)^{1/2}$$

This statement has been proven in [28] in the internal case. Later on, it was extended to the boundary case in [19], [20].

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